Speakable in quantum mechanics

Ronnie Hermens

University of Groningen

October 10, 2012

Background (old) problem: quantum probabilities

- Born rule:

$$\mathrm{Prob}_{\rho}(A \in \Delta) = \mathrm{Tr}(\rho P_A^{\Delta}),$$

  with P_A PVM associated with A.
- Simplicity assumption: A is a self-adjoint operator on a finite dimensional Hilbert space.
- What are quantum probabilities probabilities of?
- Assumption: probability that the proposition $A \in \Delta$ is true.
- What does the proposition express and what is the logic of such propositions?

Proposals for $A \in \Delta$

- Realist conjunction: "A has a value and it lies in the set Δ."
- Instrumentalist conjunction: "A is measured and the result lies in Δ."
- Conditional: "If A is measured, then the result lies in Δ."'

All face problems with orthodox quantum logic. Consider $A_1, A_2, \Delta_1, \Delta_2$ such that

$$P_{A_1}^{\Delta_1} = P_{A_1}^{\Delta_1} \wedge \left(P_{A_2}^{\Delta_2} \vee P_{A_2}^{\Delta_2^c}\right) \neq \left(P_{A_1}^{\Delta_1} \wedge P_{A_2}^{\Delta_2}\right) \vee \left(P_{A_1}^{\Delta_1} \wedge P_{A_2}^{\Delta_2^c}\right) = 0.$$

None of the readings can explain both equalities.
Options:

(a) None of these propositions capture $A \in \Delta$.

(b) The propositions of the form $A \in \Delta$ are not fully characterized by projections.

Motivation (new problem)

- If orthodox quantum logic does not describe these propositions, which logic does?
- Program: set up a new quantum logic based on propositions of the form

$$M_A(\Delta) = \text{“}A \text{ is measured and the result lies in } \Delta.\text{”}$$

- Start with the instrumentalist conjunctions in stead of the other propositions because:
  1. Neutral with respect to the interpretation of the theory.
  2. Conditionals are difficult.
  3. Conjunctions seem more primitive and may be used to define conditionals.

A preorder for elementary propositions

- Total set of elementary propositions:

$$EP_{QM} = \{M_A(\Delta)\,;\; A = A^*, \Delta \subset \mathrm{Spec}(A)\}.$$

- **LMR** (Law-Measurement Relation):
  If $A_2 = f(A_1)$, then $M_{A_1}(\Delta_1)$ implies $M_{A_2}(f(\Delta_1))$.
- Leads to the preorder

$$M_{A_1}(\Delta_1) \leq M_{A_2}(\Delta_2) \text{ iff } \mathscr{Alg}(A_1) \supset \mathscr{Alg}(A_2) \text{ and } P^{\Delta_1}_{A_1} \leq P^{\Delta_2}_{A_2}$$

- **IEA** (Idealized Experimenter Assumption):
  Every measurement has an outcome ($M_A(\varnothing) = \bot$).
- Leads to the preorder

$$M_{A_1}(\Delta_1) \leq M_{A_2}(\Delta_2) \text{ iff } \mathscr{Alg}(A_1) \supset \mathscr{Alg}(A_2) \text{ and } P^{\Delta_1}_{A_1} \leq P^{\Delta_2}_{A_2} \text{ or } P^{\Delta_1}_{A_1} = 0.$$

The logic generated by elementary propositions

- The lattice of elementary propositions is

$$EP_{QM}/\sim \simeq S_{QM} := \left\{ (\mathcal{A}, P)\,;\, {\mathcal{A}\text{ Abelian algebra,} \atop P = P^* = P^2 \in \mathcal{A},\ P \neq 0} \right\} \cup \{\bot\}.$$

- Introducing disjunctions and conjunctions require extending the lattice to

$$L_{QM} := \left\{ S : \mathfrak{A} \to \mathrm{Proj}(\mathcal{H})\,;\, {S(\mathcal{A}) \in \mathcal{A} \atop S(\mathcal{A}_1) \leq S(\mathcal{A}_2)\text{ whenever } \mathcal{A}_1 \subset \mathcal{A}_2} \right\}$$

where the embedding is given by

$$(\mathcal{A}, P) \mapsto S_{(\mathcal{A},P)},\ S_{(\mathcal{A},P)}(\mathcal{A}') = \begin{cases} P & \mathcal{A} \subset \mathcal{A}', \\ 0 & \text{else.} \end{cases}$$

The logic generated by elementary propositions

- $L_{QM} := \left\{ S : \mathfrak{A} \to \mathrm{Proj}(\mathcal{H})\,;\ {}^{S(\mathcal{A})\in\mathcal{A}}_{S(\mathcal{A}_1)\leq S(\mathcal{A}_2) \text{ whenever } \mathcal{A}_1\subset\mathcal{A}_2} \right\}$ is a Heyting algebra with the operations

$$\begin{aligned}(S_1 \vee S_2)(\mathcal{A}) &= S_1(\mathcal{A}) \vee S_2(\mathcal{A}),\\ (S_1 \wedge S_2)(\mathcal{A}) &= S_1(\mathcal{A}) \wedge S_2(\mathcal{A}),\\ (S_1 \to S_2)(\mathcal{A}) &= \bigwedge \left\{ S_1(\mathcal{A}')^\perp \vee S_2(\mathcal{A}')\,;\ {}^{\mathcal{A}'\supset\mathcal{A}}_{S_1(\mathcal{A}')^\perp\vee S_2(\mathcal{A}')\in\mathcal{A}} \right\}.\end{aligned}$$

- This logic is 'old' and first occurred in (Caspers,Heunen,Landsman,Spitters 2009).

Some properties of L_{QM}

- Special case of the conjunction:

$$S_{(\mathcal{A}_1,P_1)} \wedge S_{(\mathcal{A}_2,P_2)} = \begin{cases} S_{(\mathcal{Alg}(\mathcal{A}_1,\mathcal{A}_2),P_1\wedge P_2)} & \text{if } [\mathcal{A}_1,\mathcal{A}_2]=0, \\ \perp & \text{else,} \end{cases}$$

- Special case of the disjunction:

$$S_{(\mathcal{A},P_1)} \vee S_{(\mathcal{A},P_2)} = S_{(\mathcal{A},P_1\vee P_2)}.$$

- "If A is measured, then the result lies in Δ" can be associated with

$$\left(S_{(\mathcal{A},1)} \to S_{(\mathcal{A},P_A^\Delta)}\right)(\mathcal{A}') = \begin{cases} 1 & [\mathcal{A},\mathcal{A}'] \neq 0, \\ P & [\mathcal{A},\mathcal{A}'] = 0, P \in \mathcal{A}', \\ 0 & [\mathcal{A},\mathcal{A}'] = 0, P \notin \mathcal{A}'. \end{cases}$$

- Peculiar property:

$$\left(S_{(\mathcal{A},1)} \to S_{(\mathcal{A},P_A^\Delta)}\right) \vee \left(S_{(\mathcal{A},1)} \to S_{(\mathcal{A},P_A^{\Delta^c})}\right) = S_{(\mathcal{Alg}(P),1)} \vee \neg S_{(\mathcal{A},1)} < \top$$

Return to the background problem

- Can there be probability functions $\mathrm{Prob} : L_{QM} \to [0,1]$ such that

$$\mathrm{Prob}_\rho\left(S_{(\mathcal{A},P)}\right) = \mathrm{Tr}(\rho P)$$

or

$$\mathrm{Prob}_\rho\left(S_{(\mathcal{A},1)} \to S_{(\mathcal{A},P)}\right) = \mathrm{Tr}(\rho P)?$$

- Answer: No. At least, not without running into conflict with the interpretation of the elements of L_{QM}.
  So these propositions do not capture $A \in \Delta$.
- Further options:
  1. Quantum probabilities are conditional probabilities (and not probabilities of conditionals):

$$\mathrm{Prob}_\rho\left(S_{(\mathcal{A},P)} | S_{(\mathcal{A},1)}\right) = \mathrm{Tr}(\rho P)?$$

  2. L_{QM} is not the full quantum logic: e.g., the relative pseudo-complement does not correspond to the conditional.

The end (?)

Thank You