Placing Probabilities of Conditionals in Context

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Motto

Van Fraassen (1976):

> *"What is the probability that I throw a six if I throw an even number, if not the probability that: if I throw an even number, it will be a six?" [5]*

$$\textbf{Thesis:}\ \mathbb{P}_{\text{even}}(\text{six}) = \mathbb{P}(\text{even} \rightarrow \text{six})$$

More on the safe side:

- Can the updated (posterior) probability of B be understood as the *prior* probability of some proposition "A thing B"?

$$\exists \ltimes \text{ such that } \mathbb{P}_A(B) = \mathbb{P}(A \ltimes B)?$$

I.e., is there a connective for which the Thesis holds?

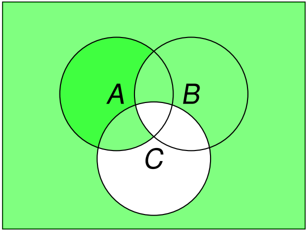
- Additional goal: highlight and question slumbering background assumptions.

Overview

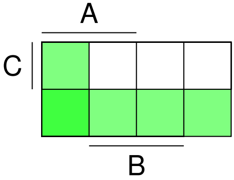
Karnaugh maps

Pictures are nice, but nice pictures are nicer

A Venn diagram:



A Karnaugh map:



Displaying propositions:

$$(A \wedge \neg B) \vee \neg C$$

Karnaugh maps

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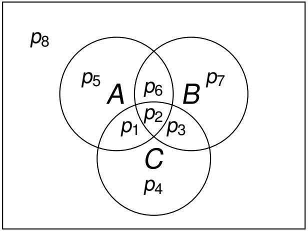
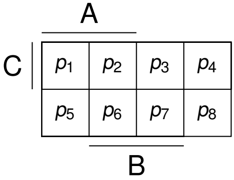


A Karnaugh map:



Displaying propositions:

$$(A \wedge \neg B) \vee \neg C$$

Specifying probability functions:

$$p_1 + \cdots + p_8 = 1,\ 0 \leq p_i \leq p_0.$$

Updating probabilities

An important rule for updating is the traditional Bayes' rule:

$$\mathbb{P}_A(B) \text{ (updated probability)} = \mathbb{P}(B|A) \text{ (conditional probability)}$$

$$\text{where } \mathbb{P}(B|A) \stackrel{\text{def}}{=\!=} \frac{\mathbb{P}(A \wedge B)}{\mathbb{P}(A)}, \qquad (\mathbb{P}(A) > 0)$$

A (columns 1–2), B (columns 2–3), C (row 1):

p_1	p_2	p_3	p_4
p_5	p_6	p_7	p_8

$\Longrightarrow$

A (columns 1–2), B (columns 2–3), C (row 1):

$\frac{p_1}{\mathbb{P}(A)}$	$\frac{p_2}{\mathbb{P}(A)}$	0	0
$\frac{p_5}{\mathbb{P}(A)}$	$\frac{p_6}{\mathbb{P}(A)}$	0	0

! This is a (motivated) **assumption** and not to be confused with Bayes' **theorem**:

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B)}{\mathbb{P}(A)} \mathbb{P}(A|B)$$

The extent to which Bayes' rule holds will be a point of discussion.

The Thesis as a constraint on probability functions

- Without any constraints the possibilities for the probabilities for $\ltimes$ are

A

$A \ltimes B$				
	p_1	p_2	p_3	p_4
	p_5	p_6	p_7	p_8

B

- With Bayes' rule one has

$$p_1 + p_2 + p_3 + p_4 = \mathbb{P}(A \ltimes B) = \mathbb{P}_A(B) = \mathbb{P}(B|A) = \frac{p_2 + p_6}{p_1 + p_2 + p_5 + p_6}$$

- Although this does not look very restrictive (8 variables, 1 equation) there turn out to be interesting and possibly disturbing consequences.

Some first insights

- The Thesis requires that $\ltimes$ is not the material implication $\supset$.

A (columns 1–2); B (columns 2–3); $A \supset B$ (rows 1–2); $A \supset \neg B$ (rows 2–3)

$\emptyset$	p_2	$\emptyset$	$\emptyset$
$\emptyset$	$\emptyset$	p_7	p_8
p_9	$\emptyset$	$\emptyset$	$\emptyset$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

$$\begin{array}{r} \mathbb{P}(A \supset B) + \mathbb{P}(A \supset \neg B) = 1 + \mathbb{P}(\neg A) \\ \mathbb{P}(B|A) + \quad \mathbb{P}(\neg B|A) = 1 \\ \hline 0 = \mathbb{P}(\neg A) \end{array}$$

More generally, $\ltimes$ is *not* truth-functional.

- In extremal cases $\ltimes$ *does* behave truth-functional:

When $\mathbb{P}(A) > 0$:

$$\mathbb{P}(A \ltimes B) = 1 \text{ iff } \mathbb{P}(A \supset B) = 1$$
$$\mathbb{P}(A \ltimes B) = 0 \text{ iff } \mathbb{P}(A \wedge B) = 0$$

Lewis' triviality result [2]

- If a probability function $\mathbb{P}$ satisfies the Thesis and Bayes' rule for some connective $A \ltimes B$, then $\mathbb{P}(B|A) = \mathbb{P}(B)$. ☹
- Don't like the result? Investigate the proof.

Three steps

1. Apply the Thesis and Bayes' rule: (Assumption)
$$\mathbb{P}(B|A) = \mathbb{P}_A(B) = \mathbb{P}(A \ltimes B)$$
2. Apply the law of total probability: (Theorem)
$$\mathbb{P}(A \ltimes B) = \mathbb{P}(A \ltimes B|B)\,\mathbb{P}(B) + \mathbb{P}(A \ltimes B|\neg B)\,\mathbb{P}(\neg B)$$
3. Apply the Thesis and Bayes' rule to the conditionalized probabilities:(?)
$$\mathbb{P}(A \ltimes B|B) = \mathbb{P}_A(B|B) = \mathbb{P}(B|A \wedge B) = 1$$
$$\mathbb{P}(A \ltimes B|\neg B) = \mathbb{P}_A(B|\neg B) = \mathbb{P}(B|A \wedge \neg B) = 0$$

Scrutinizing step 3

Step three uses that the conditionalized probability function $\mathbb{P}(.|B)$ is a probability function that satisfies the Thesis.
Two assumptions:

1. $\mathbb{P}_B$ is again a probability function that satisfies the Thesis.
2. $\mathbb{P}_B(.)$ behaves as $\mathbb{P}(.|B)$ on all propositions (e.g. also those containing $\ltimes$).

Bayes' rule 2 relies on the idea that after the update the 'interpretation' of propositions is the same as before the update:

$$\text{Bayes' rule} \Leftrightarrow \frac{\mathbb{P}_B(A)}{\mathbb{P}_B(\neg A)} = \frac{\mathbb{P}(B \wedge A)}{\mathbb{P}(B \wedge \neg A)}$$

For 'simple' propositions this seems natural.
Why should this be assumed to hold for $\ltimes$ if we do not know yet what $\ltimes$ is?

Why should Bayes' rule fail. . .

- . . . for conditionals? Van Fraassen:

  "[Lewis assumes that it] should be possible to make this revision by changing the probability measure alone – and not the constitution of the possible worlds" [5]

Possible worlds and probabilities before updating on B:

A

A→B

p_1	p_2	p_3	p_4
p_5	p_6	p_7	p_8

B

Possible worlds and probabilities after updating on B:

A

A→B

0	p'_2	p'_3	0
0	p'_6	p'_7	0

B

- . . . for $\ltimes$? As *the* connective that satisfies the Thesis, this shift towards contextuality is rather natural: the Thesis itself suggests that the meaning of $\ltimes$ depends on $\mathbb{P}$.

Reversing Lewis' proof

Three assumptions:

1. The Thesis: there is a connective $\ltimes$ such that for every $\mathbb{P}$, $\mathbb{P}(A \ltimes B) = \mathbb{P}_A(B)$.
2. Bayes' minimal rule: for all propositions A and B that do not contain $\ltimes$, $\mathbb{P}_A(B) = \mathbb{P}(B|A)$.
3. Consecutive updating is updating on conjunctions: $(\mathbb{P}_A)_B = \mathbb{P}_{A\wedge B}$.

Imply an update rule for $\ltimes$:

$$\mathbb{P}_A(B \ltimes C) \overset{1}{=} (\mathbb{P}_A)_B(C) \overset{3}{=} \mathbb{P}_{A\wedge B}(C) \overset{2}{=} \mathbb{P}(C|A \wedge B)$$

Lewis' result can now be understood as showing that this rule does not coincide with Bayes' rule

$$\mathbb{P}_A(B \ltimes C) \neq \mathbb{P}(B \ltimes C|A)$$

But not as a result against the Thesis.

Conditional-like properties of $\ltimes$

- The Thesis suggests Modus Ponens:

$$\mathbb{P}(A) = 1,\ \mathbb{P}(A \ltimes B) = 1 \ \Rightarrow\ \mathbb{P}(B) = 1$$

- And also for nested occurrences:

$$\mathbb{P}(A) = 1,\ \mathbb{P}(A \ltimes (B \ltimes C)) = 1 \ \Rightarrow\ \mathbb{P}(B \ltimes C) = 1$$

- The Thesis suggests Import-Export:

$$\mathbb{P}(A \ltimes (B \ltimes C)) = \mathbb{P}((A \wedge B) \ltimes C)$$

From now on $\ltimes$ will be investigated as a conditional $\rightarrow$

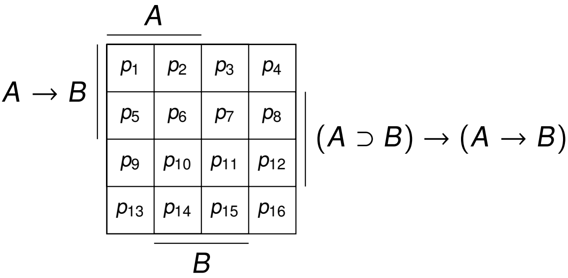
Obstacles:

- Conjunctions satisfy the same properties while being quite distinct from conditionals.
- The Thesis implies that if $A \models B$, then $\mathbb{P}(A \rightarrow B)$=1. Then, according to Gibbard [1], $\rightarrow$ must be the material implication $\supset$.

Gibbard's problem for the Thesis

Theorem: There is no connective $\rightarrow$ that satisfies the Thesis, Import-Export and Modus Ponens

Proof:



Gibbard's problem for the Thesis

Theorem: There is no connective $\rightarrow$ that satisfies the Thesis, Import-Export and Modus Ponens

Proof:

A (columns 1–2)

$A \rightarrow B$ (rows 1–2)

$\varnothing$	p_2	p_3	p_4
$\varnothing$	p_6	p_7	p_8
p_9	p_{10}	p_{11}	p_{12}
p_{13}	p_{14}	p_{15}	p_{16}

$(A \supset B) \rightarrow (A \rightarrow B)$ (rows 2–3)

B (columns 2–3)

Gibbard's problem for the Thesis

Theorem: There is no connective $\rightarrow$ that satisfies the Thesis, Import-Export and Modus Ponens

Proof:

A $\quad A \supset B$

$A \rightarrow B$

$\emptyset$	p_2	p_3	p_4
$\emptyset$	p_6	p_7	p_8
p_9	$\emptyset$	$\emptyset$	$\emptyset$
p_{13}	p_{14}	p_{15}	p_{16}

$(A \supset B) \rightarrow (A \rightarrow B)$

B

Gibbard's problem for the Thesis

Theorem: There is no connective $\rightarrow$ that satisfies the Thesis, Import-Export and Modus Ponens

Proof:

Columns: A (columns 1–2), $A \supset B$ (columns 2–4), B (columns 2–3); rows: $A \rightarrow B$ (rows 1–2), $(A \supset B) \rightarrow (A \rightarrow B)$ (rows 2–3)

$\varnothing$	0	0	0
$\varnothing$	p_6	p_7	p_8
p_9	$\varnothing$	$\varnothing$	$\varnothing$
0	0	0	0

Import-Export: $(A \supset B) \rightarrow (A \rightarrow B) = (A \wedge B) \rightarrow B$.

$\Rightarrow$ Thus $\rightarrow$ coincides with $\supset$: this is in contradiction with the Thesis!

What to do?

- McGee [4]: save the Thesis, discard Modus Ponens (for nested conditionals). Counterexample [3]:

Opinion polls taken just before the 1980 election showed the Republican Ronald Reagan decisively ahead of the Democrat Jimmy Carter, with the other Republican in the race, John Anderson, a distant third. Those apprised of the poll results believed, with good reason:

- If a Republican wins the election, then if it's not Reagan who wins it will be Anderson.
- A Republican will win the election.

Yet they did not have reason to believe

- If it's not Reagan who wins, it will be Anderson.

This reads as a counterexample to the logical inference:

$$\frac{(A \vee R) \rightarrow (\neg R \rightarrow A) \qquad A \vee R}{\neg R \rightarrow A}$$

Is McGee's counter example compatible with the Thesis?

- McGee's example suggests that

$$\mathbb{P}((A \vee R) \to (\neg R \to A)) \text{ and } \mathbb{P}(A \vee R) \text{ are high,}$$
$$\text{while } \mathbb{P}(\neg R \to A) \text{ is low}$$

- A calculation using the Thesis gives

$$\mathbb{P}(\neg R \to A) = 1 - \frac{\mathbb{P}(C)}{\mathbb{P}(A) + \mathbb{P}(C)}$$

1. If $\mathbb{P}(A \vee R) = 1$, then $\mathbb{P}(C) = 0$ and $\mathbb{P}(\neg R \to A) = 1$.
2. If $0 < \mathbb{P}(C) \ll \mathbb{P}(A)$, then $\mathbb{P}(\neg R \to A) \simeq 1$.
3. If $\mathbb{P}(C) \gg \mathbb{P}(A)$, then $\mathbb{P}(\neg R \to A) \ll 1$.

- The Thesis is compatible, but also suggest that this isn't a counterexample to Modus Ponens (but more akin to the lottery paradox).

What is going on?

- The tension at hand is more peculiar than McGee would have us believe. On the one hand the Thesis seems to suggest MP, and on the other hand the two seem incompatible (Gibbards theorem).
- Don't like the result? Investigate the proof.

Different notions of Modus Ponens:

1. Probabilistic: $\mathbb{P}(A) = 1$, $\mathbb{P}(A \to B) = 1$, then $\mathbb{P}(B) = 1$.
2. Set-theoretic: $[\![A]\!] \cap [\![A \to B]\!] \subset [\![B]\!]$.

2 (used in the proof) implies 1 (a derived property of $\ltimes$) but the converse is not true.

- Remember with van Fraassen that $[\![A \to B]\!]$ need not stay fixed.
- It is sufficient for 1 if 2 only holds in the case where $\mathbb{P}(A) = 1$.

Going contextual (1)

Two questions:

- How can we keep track of the changing of the set of possible worlds associated with a conditional?
- Is this contextual take sufficient to counter the proof?

Make the context $\mathfrak{C}$ explicit in the notation. This makes clear that in general

$$\left[\!\left[A \xrightarrow{\mathfrak{C}} B \right]\!\right] \neq \left[\!\left[A \xrightarrow{\mathfrak{C}'} B \right]\!\right]$$

The Thesis with Bayes' rule can now be written as

$$\mathbb{P}_{\mathfrak{C}}(A \xrightarrow{\mathfrak{C}} B) = \mathbb{P}_{\mathfrak{C}_A}(B) = \mathbb{P}_{\mathfrak{C}}(B|A)$$

And for Modus Ponens

$$\mathbb{P}_{\mathfrak{C}}(A) = \mathbb{P}_{\mathfrak{C}}(A \xrightarrow{\mathfrak{C}} B) = 1, \text{ then } \mathbb{P}_{\mathfrak{C}}(B) = 1,$$

$$\mathbb{P}_{\mathfrak{C}_A}(A \xrightarrow{\mathfrak{C}_A} B) = 1, \text{ then } \mathbb{P}_{\mathfrak{C}_A}(B) = 1$$

Going contextual (2)

- The proof assumed that $[\![A]\!] \cap \left[\!\!\left[A \overset{\mathfrak{C}}{\longrightarrow} B\right]\!\!\right] \subset [\![B]\!]$ for every $\mathfrak{C}$ *independent* of the values of the p_i (i.e., independent of $\mathbb{P}_{\mathfrak{C}}$).

Column labels: A (columns 1–2); $A \supset B$ (columns 2–4); B (columns 2–3, below). Row labels: $A \overset{\mathfrak{C}}{\longrightarrow} B$ (rows 1–2); $(A \supset B) \overset{\mathfrak{C}}{\longrightarrow} (A \overset{\mathfrak{C}}{\longrightarrow} B)$ (rows 2–3).

$\emptyset$	0	0	0
$\emptyset$	p_6	p_7	p_8
p_9	$\emptyset$	$\emptyset$	$\emptyset$
0	0	0	0

- The proof assumed that $[\![A]\!] \cap \left[\!\!\left[A \overset{\mathfrak{C}}{\longrightarrow} B\right]\!\!\right] \subset [\![B]\!]$ for every $\mathfrak{C}$ *independent* of the values of the p_i (i.e., independent of $\mathbb{P}_{\mathfrak{C}}$).

Columns: A (columns 1–2); $A \supset B$ (columns 2–4); B (columns 2–3, below)

Rows: $A \overset{\mathfrak{C}}{\longrightarrow} B$ (rows 1–2); $(A \supset B) \overset{\mathfrak{C}}{\longrightarrow} (A \overset{\mathfrak{C}}{\longrightarrow} B)$ (rows 2–3)

0	0	0	0
p_5	p_6	p_7	p_8
p_9	p_{10}	p_{11}	p_{12}
0	0	0	0

- But for the Thesis the available options are richer.

- The proof assumed that $[\![A]\!] \cap \left[\!\!\left[A \overset{\mathfrak{C}}{\rightarrow} B \right]\!\!\right] \subset [\![B]\!]$ for every $\mathfrak{C}$ *independent* of the values of the p_i (i.e., independent of $\mathbb{P}_{\mathfrak{C}}$).

A (columns 1–2); $A \supset B$ (columns 2–4)

$A \xrightarrow{\mathfrak{C}_{A\supset B}} B$ (rows 1–2)

0	0	0	0
0	p_6	p_7	p_8
0	$\emptyset$	$\emptyset$	$\emptyset$
0	0	0	0

$(A \supset B) \xrightarrow{\mathfrak{C}_{A\supset B}} (A \xrightarrow{\mathfrak{C}_{A\supset B}} B)$ (rows 2–3)

B (columns 2–3)

- But for the Thesis the available options are richer.
- Restrictions only hold for contexts of the form $\mathfrak{C}_{A\supset B}$.

Short summary

Assumptions:

- The Thesis: $\mathbb{P}_{\mathfrak{C}}(A \overset{\mathfrak{C}}{\rightarrow} \phi) = \mathbb{P}_{\mathfrak{C}_A}(\phi)$.
- Bayes' rule when ϕ does not contain $\overset{\mathfrak{C}}{\rightarrow}$: $\mathbb{P}_{\mathfrak{C}_A}(\phi) = \mathbb{P}_{\mathfrak{C}}(\phi|A)$.
- Consecutive updating is updating on the conjunction: $\mathbb{P}_{(\mathfrak{C}_A)_B}(\phi) = \mathbb{P}_{\mathfrak{C}_{A\wedge B}}(\phi)$.

Consequences:

- Alternate update rule for conditionals: $\mathbb{P}_{\mathfrak{C}_A}(B \overset{\mathfrak{C}_A}{\longrightarrow} \phi) = \mathbb{P}_{\mathfrak{C}}(\phi|A \wedge B) \neq \mathbb{P}_{\mathfrak{C}}(B \overset{\mathfrak{C}}{\rightarrow} \phi|A)$
- Probabilistic Import-Export: $\mathbb{P}_{\mathfrak{C}}(A \overset{\mathfrak{C}}{\rightarrow} (B \overset{\mathfrak{C}}{\rightarrow} \phi)) = \mathbb{P}_{\mathfrak{C}}((A \wedge B) \overset{\mathfrak{C}}{\rightarrow} \phi)$
- Probabilistic Modus Ponens: $\mathbb{P}_{\mathfrak{C}}(A) = 1$, $\mathbb{P}_{\mathfrak{C}}(A \overset{\mathfrak{C}}{\rightarrow} \phi) = 1$, then $\mathbb{P}_{\mathfrak{C}}(\phi) = 1$.
- Contextuality: $\left[\!\left[A \overset{\mathfrak{C}}{\rightarrow} \phi \right]\!\right]$ depends on $\mathfrak{C}$.

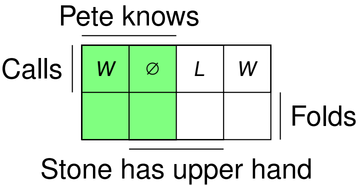
Putting contextuality to work

- Not everybody likes the idea of contextuality:
  "presumably our indicative conditional has a fixed interpretation, the same for speakers with different beliefs, and for one speaker before and after a change in his beliefs. Else how are disagreements about a conditional possible, or changes of mind?" - Lewis [2]
- Making the conditional contextual is not merely a trick to uphold the Thesis. It may actually make sense.

Consider Gibbards [1] story on the riverboat:

> *"Sly Pete and Mr. Stone are playing poker on a Mississippi riverboat. It is now up to Pete to call or fold. My henchman Zack sees Stone's hand [. . .] and signals its contents to Pete. My henchman Jack sees both hands, and sees that [. . .] Stone's hand is the winning hand. [. . .] A few minutes later, Zack slips me a note which says "If Pete called, he won," and Jack slips me a note which says "If Pete called, he lost.""*

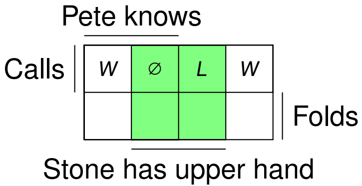
Sly Pete 2



Zack: $\mathbb{P}_{\mathfrak{C}_P}(C \xrightarrow{\mathfrak{C}_P} W) = \mathbb{P}_{\mathfrak{C}}(P \xrightarrow{\mathfrak{C}} (C \xrightarrow{\mathfrak{C}} W))$

$$= \mathbb{P}_{\mathfrak{C}}((P \wedge C) \xrightarrow{\mathfrak{C}} W) = \mathbb{P}_{\mathfrak{C}}(W | P \wedge C) = 1$$

Sly Pete 2



$$\text{Zack: } \mathbb{P}_{\mathfrak{C}_P}(C \xrightarrow{\mathfrak{C}_P} W) = \mathbb{P}_{\mathfrak{C}}(P \xrightarrow{\mathfrak{C}} (C \xrightarrow{\mathfrak{C}} W))$$

$$= \mathbb{P}_{\mathfrak{C}}((P \wedge C) \xrightarrow{\mathfrak{C}} W) = \mathbb{P}_{\mathfrak{C}}(W|P \wedge C) = 1$$

$$\text{Jack: } \mathbb{P}_{\mathfrak{C}_S}(C \xrightarrow{\mathfrak{C}_S} L) = \mathbb{P}_{\mathfrak{C}}(S \xrightarrow{\mathfrak{C}} (C \xrightarrow{\mathfrak{C}} L))$$

$$= \mathbb{P}_{\mathfrak{C}}((S \wedge C) \xrightarrow{\mathfrak{C}} L) = \mathbb{P}_{\mathfrak{C}}(L|S \wedge C) = 1$$

- Both henchmen are right, but assign different probabilities to the conditional because they evaluate it in different contexts.

Questions?

1. Although Modus Ponens is satisfied in some sense, the set of possible worlds $[\![A]\!] \cap [\![A \to \phi]\!] \cap [\![\neg\phi]\!]$ still has probability greater than zero. How to make sense of this?
2. Supposing the set of possible worlds associated with a conditional depends on a persons epistemic state, in what sense does the conditional express a proposition?
3. How exactly does/should the set of possible worlds associated with a conditional change when the context is updated?
4. And how about updating on a conditional?
5. Can the Thesis shed any light on the distinction between indicative conditionals and counterfactuals?
6. Is the Thesis true?

Some Useful Sources I

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