Quantum Logic & Quantum Probability
An Empiricist Approach

Ronnie Hermens

University of Groningen

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Outline 1.0

1. Introduction
2. Outline 2.0

Primitive concepts in QM

Born Postulate

If

1. *the state of a quantum system is represented by the density operator* ρ *and*
2. *a measurement of an observable* $\mathcal{A}$ *associated with the operator* A *is performed,*

then the probability to obtain a result in the set $\Delta \subset \mathbb{R}$ *is given by*

$$\mathrm{Tr}\left(\rho\mu_A(\Delta)\right)$$

with μ_A *the PVM associated with* A*.*

"The concept of 'measurement' becomes so fuzzy on reflection that it is quite surprising to have it appearing in physical theory at the most fundamental level..." - Bell 1981

...And so for 'probability'...

Reconstructing (a part of) quantum mechanics

Can the Born rule be derived from the other postulates of quantum theory with the aid of a conceptual assumptions on what probability is?

- Find an X such that $\mathrm{QM}\backslash\mathrm{BR} \wedge \mathrm{X} \rightarrow \mathrm{BR}$.
- Find necessary aspects of X.

Bottom up:

- Introduce ontology by solving measurement problem.
- Derive BR.

Top down:

- Construct formal framework that captures empirical aspects of QM. (Quantum Logic)
- Introduce conceptual notion of probability. Derive BR.

Outline 2.0

1. Introduction. ✓
2. The violation of Bell inequalities.
3. Problems with orthodox quantum logic.
4. An empiricist quantum logic.
5. What is needed to get the Born rule?

How does QM violate Bell inequalities?

- Source Independence: $\rho_{M_A,M_B}(\lambda) = \rho(\lambda)$
- Parameter Independence: $P_{M_A,M_B}(A = i|\lambda) = P_{M_A}(A = i|\lambda)$
- Outcome Independence:
  $P_{M_A,M_B}(A = i|B = j, \lambda) = P_{M_A,M_B}(A = i|\lambda)$

$$P_{M_{A_1},M_{B_1}}(A_1 = B_1) \leq P_{M_{A_1},M_{B_2}}(A_1 = B_2) + P_{M_{A_2},M_{B_1}}(A_2 = B_1) + P_{M_{A_2},M_{B_2}}(A_2 = B_2)$$

Earman (1986): "in the first instance, the issue of locality is a red herring. [. . .] The impossibility emerges from the X – the existence of a phase space representation – whether or not Nature operates locally or at a distance."
Fine (1982): Bell's theorem imposes "requirements to make well defined precisely those probability distributions for noncommuting observables whose rejection is the very essence of quantum mechanics"

How does QM violate Bell inequalities? 2

Bell inequality as a 'logical' consequence:

$$P(A_1 \wedge B_1) \leq P(A_1 \wedge B_2) + P(A_2 \wedge B_1) + P(\neg A_2 \wedge \neg B_2)$$

Proof:

$$\begin{aligned} P(A_1 \wedge B_1) =& P(A_1 \wedge B_1 \wedge (B_2 \vee \neg B_2)) \\ =& P((A_1 \wedge B_1 \wedge B_2) \vee (A_1 \wedge B_1 \wedge \neg B_2)) \\ =& P(A_1 \wedge B_1 \wedge B_2) + P(A_1 \wedge B_1 \wedge \neg B_2) \\ \leq& P(A_1 \wedge B_2) + P(A_1 \wedge B_1 \wedge \neg B_2) \\ \leq& \ldots \leq P(A_1 \wedge B_2) + P(A_2 \wedge B_1) + P(\neg A_2 \wedge \neg B_2) \end{aligned}$$

-) Putnam (1968): in quantum logic distributivity is not valid.
-) Griffiths (2013): new quantum logic has "single framework rule".
-) Khrennikov (2015): derivation assumes Kolmogorov's probability.

-) Hermens (2008): derivation not valid in intuitionistic logic.

Quantum logic and the Born rule

A success story

Suppose

- The lattice of projection operators provides the correct 'logic' for QM,

and

- One finds an argument to motivate $P(1) = 1$ and

$$P(P_1 \vee P_2) = P(P_1) + P(P_2)$$

whenever $P_1 \perp P_2$,

then

- Gleason's Theorem states P is a probability function iff it satisfies the Born rule.

Modern version of the success story

Wilce (2012): "If we put aside scruples about 'measurement' as a primitive term in physical theory, and accept a principled distinction between 'testable' and non-testable properties, then the fact that $L(\mathcal{H})$ is not Boolean is unremarkable"

Bub (2007): "The rejection of both dogmas, as I will argue in the following section, leads to an information-theoretic interpretation of quantum mechanics. On this interpretation, the structure of Hilbert space, i.e., the non-Boolean algebra of Hilbert space subspaces, defines the structure of a quantum event space"

- Why are the (closed) linear subspaces worthy of the name "event" or " property"?
- The primitive use of "probability" has merely been replaced.

Outline 2.7

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Revisiting Birkhoff and von Neumann

"The object of the present paper is to discover what logical structure one may hope to find in physical theories which, like quantum mechanics, do not conform to classical logic" - BvN 1936

- **Method:** establishing a connection between "experimental propositions" that live in "observation-spaces" and subsets of the "phase-space".
- **Phase-space:** This is the Hilbert space $\mathcal{H}$.
- **Observation-space:** Let $A_1, \ldots, A_n$ be compatible observables with spectra $\sigma(A_1), \ldots, \sigma(A_n)$, then the corresponding observation-space is

$$\sigma(A_1) \times \ldots \times \sigma(A_n),$$

the set of possible outcomes.

Revisiting BvN 2: Establishing a connection

How to establish a connection...

- **Definition:** The *mathematical representative* of an experimental proposition $\Delta \subset \sigma(A_1) \times \ldots \times \sigma(A_n)$ is the set of states in $\mathcal{H}$ for which the probability of finding a result in Δ given a measurement of $A_1, \ldots, A_n$ equals 1.
- Simple case $n = 1$:

$$\sigma(A) \supset \Delta \mapsto \mu_A(\Delta)\mathcal{H}$$

  with μ_A the PVM associated with A.
- More generally, these are the states in the subspace

$$\left(\bigvee_{\{(a_1,\ldots,a_n)\in\Delta\}} \bigwedge_{i=1}^{n} \mu_{A_i}(\{a_i\}) \right) \mathcal{H}.$$

Revisiting BvN 3: The whole story?

$$\begin{array}{ccc} \mathcal{P}(\sigma(A_1) \times \ldots \times \sigma(A_n)) & \longrightarrow & L(\mathcal{H}) = \{P : \mathcal{H} \to \mathcal{H} | P = P^* = P^2\} \\ & \nearrow & \\ \mathcal{P}(\sigma(B_1) \times \ldots \times \sigma(B_m)) & & \end{array}$$

- The association defines for every observation space a lattice homomorphism taking experimental propositions to projection operators.
- Running over all observation spaces one ranges over the entirety of $L(\mathcal{H})$.
- Does $L(\mathcal{H})$ give the logic of all experimental propositions?
- Two background assumptions can be identified for getting a "yes":
  1. It is unproblematic to 'forget the measurement context' when correlating an experimental proposition to the phase-space.
  2. Disjunctions, conjunctions and negations of experimental propositions are again experimental propositions.

What is an experimental proposition?

- BvN don't go deep into this question, but at least seem to assume that it is a proposition that can serve as a prediction and can be tested.
- Inspiration from Bohr (1948):

  "all well-defined experimental evidence, even if it cannot be analyzed in terms of classical physics, must be expressed in ordinary language making use of common logic"

- Experimental propositions should thus be expressible in ordinary language.
- What kind of expressions would fit well with the program of BvN?

What is an experimental proposition? 2

- When considering a single observation-space the observation itself is presupposed.
- Example: $\sigma(A) = \{0, 1\}$, then $\mu_A(0) \vee \mu_A(1)$ is considered a tautology.
- These presuppositions seem to be neglected when considering multiple observation-spaces.
- Example: $\sigma(A) = \sigma(B) = \{0, 1\}$, $[A, B] \neq 0$

$$\mu_B(0) = \mu_B(0) \wedge (\mu_A(0) \vee \mu_A(1)) = (\mu_B(0) \wedge \mu_A(0)) \vee (\mu_B(0) \wedge \mu_A(1)) = 0.$$

The two assumptions:

1. It is unproblematic to 'forget the measurement context' when correlating an experimental proposition to the phase-space.
2. Disjunctions, conjunctions and negations of experimental propositions are again experimental propositions.

Outline 2.71

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Back to the drawing board: Empiricist quantum logic

What kind of propositions do we need?

The Born rule in a formula:

$$P(M_A(\Delta)|M_A, \rho) = \mathrm{Tr}\left(\rho\mu_A(\Delta)\right)$$

with

$$M_A = \text{“}A \text{ is measured”},$$
$$M_A(\Delta) = \text{“}A \text{ is measured and the result lies in } \Delta\text{”}.$$

- **IEA** (Idealized Experimenter Assumption):
  Every measurement has an outcome: $M_A = M_A(\sigma(A))$.

A preorder for elementary propositions

- Total set of elementary propositions:

$$EP_{QM} = \{M_A(\Delta)\,;\ A = A^*, \Delta \subset \sigma(A)\}.$$

- **LMR** (Law-Measurement Relation):
  If $A_2 = f(A_1)$, then $M_{A_1}(\Delta_1)$ implies $M_{A_2}(f(\Delta_1))$.
- Leads to the preorder

$$M_{A_1}(\Delta_1) \leq M_{A_2}(\Delta_2) \text{ iff } \mathcal{Alg}(A_1) \supset \mathcal{Alg}(A_2) \text{ and } \mu_{A_1}(\Delta_1) \leq \mu_{A_2}(\Delta_2)$$

- **IEA** (Idealized Experimenter Assumption):
  Every measurement has an outcome ($M_A(\varnothing) = \bot$).
- Leads to the preorder

$$M_{A_1}(\Delta_1) \leq M_{A_2}(\Delta_2) \text{ iff } \mathcal{Alg}(A_1) \supset \mathcal{Alg}(A_2) \text{ and } \mu_{A_1}(\Delta_1) \leq \mu_{A_2}(\Delta_2) \text{ or } \mu_{A_1}(\Delta_1) = 0.$$

A lattice of elementary propositions

$$S_{QM} := EP_{QM}/\sim = \left\{ (\mathcal{A}, P) \middle| \substack{\mathcal{A} \text{ Abelian algebra,} \\ P=P^*=P^2\in\mathcal{A},\ P\neq 0} \right\} \cup \{\perp\}.$$

Properties:

$$(\mathcal{A}_1, P_1) \leq (\mathcal{A}_2, P_2) \text{ iff } \mathcal{A}_1 \supset \mathcal{A}_2,\ P_1 \leq P_2,$$

$$(\mathcal{A}_1, P_1) \wedge (\mathcal{A}_2, P_2) = \begin{cases} (\mathcal{Alg}(\mathcal{A}_1, \mathcal{A}_2), P_1 \wedge P_2) & [\mathcal{A}_1, \mathcal{A}_2] = 0, \\ \perp & \text{else,} \end{cases}$$

$$(\mathcal{A}_1, P_1) \vee (\mathcal{A}_2, P_2) = \left(\mathcal{A}_1 \cap \mathcal{A}_2, \bigwedge \{P \in \mathcal{A}_1 \cap \mathcal{A}_2 | P \geq P_1 \vee P_2\} \right).$$

- The lattice is non-distributive.
- Disjunctions are problematic (e.g. $M_x \vee M_p = \top$).

Solution: "just add the missing propositions".

Towards empiricist quantum logic

- Find lattice $L_{QM} \supset S_{QM}$ such that $\vee$ reads as OR.
- Key observation for elements of S_{QM}:

$$(\mathcal{A}, P) = \underset{\mathcal{A}' \in \mathfrak{A}}{\mathrm{OR}} (\mathcal{A}', P'),\ P' = \begin{cases} P, & \mathcal{A}' \supset \mathcal{A} \\ 0, & \text{else.} \end{cases}$$

- and $(\mathcal{A}, P_1) \, \mathrm{OR} (\mathcal{A}, P_2) = (\mathcal{A}, P_1 \vee P_2)$.
- Then

$$(\mathcal{A}_1, P_1) \, \mathrm{OR} (\mathcal{A}_2, P_2) = \underset{\mathcal{A}' \in \mathfrak{A}}{\mathrm{OR}} (\mathcal{A}', P_1' \vee P_2'),\ P_i' = \begin{cases} P_i, & \mathcal{A}' \supset \mathcal{A}_i \\ 0, & \text{else.} \end{cases}$$

- L_{QM} is the lattice with objects of the form

$$\underset{\mathcal{A} \in \mathfrak{A}}{\mathrm{OR}} (\mathcal{A}, P_{\mathcal{A}}).$$

The empiricist quantum logic

- L_{QM} is the lattice with objects of the form

$$\mathop{\mathrm{OR}}_{\mathcal{A}\in\mathfrak{A}}(\mathcal{A}, P_{\mathcal{A}}).$$

$$L_{QM} := \{S : \mathfrak{A} \to L(\mathcal{H}) | S(\mathcal{A}) \in \mathcal{A}\}, \; S \simeq \mathop{\mathrm{OR}}_{\mathcal{A}\in\mathfrak{A}}(\mathcal{A}, S(\mathcal{A})).$$

This is a Boolean lattice with properties:

$$(S_1 \wedge S_2)(\mathcal{A}) = S_1(\mathcal{A}) \wedge S_2(\mathcal{A})$$

$$(S_1 \vee S_2)(\mathcal{A}) = S_1(\mathcal{A}) \vee S_2(\mathcal{A})$$

$$(\neg S)(\mathcal{A}) = S(\mathcal{A})^{\perp}$$

And incorporates elementary experimental propositions by the rule

$$M_A(\Delta) \mapsto S_{(\mathcal{A}lg(A), \mu_A(\Delta))}, \; S_{(\mathcal{A}, P)}(\mathcal{A}') := \begin{cases} P & \mathcal{A} \subset \mathcal{A}' \\ 0 & \text{else.} \end{cases}$$

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Probability on L_{QM}

Definition

A *conditional probability function* (c.p.f.) on L_{QM} is a collection $\{P(.|\mathcal{A})|\mathcal{A} \in \mathfrak{A}\}$ of probability functions on L_{QM} such that $P\left(S_{(\mathcal{A},1)}\middle|\mathcal{A}\right) = 1$ for all $\mathcal{A}$.

- Every density operator ρ defines a c.p.f. on L_{QM}.
- A c.p.f. P follows the Born rule iff for every $\mathcal{A}$ and P:

$$P\left(S_{(\mathcal{A}_1,P)}\middle|\mathcal{A}\right) = P\left(S_{(\mathcal{A}_2,P)}\middle|\mathcal{A}\right) \ \forall \mathcal{A}_1, \mathcal{A}_2 \supset \mathcal{A}.$$

- An assumption of non-contextuality is required to derive the Born rule.

Defending non-contextuality

NC If $\mu_{A_1}(\Delta_1) = \mu_{A_2}(\Delta_2)$, then $M_{A_1}(\Delta_1)$ and $M_{A_2}(\Delta_2)$ signify the same event.

The Born rule seems the only aspect of QM that justifies NC.

Motivating NC requires assumptions from outside QM.

Introduce some metaphysics:

- Wallace's many worlds: contextuality violates "state supervenience".
- Bub's minimalist metaphysics: quantum probabilities are Humean chances. NC = best system.

Or epistemic constraints:

- Probability functions should be continuous w.r.t. measurement outcomes and setups. + something.
- Other clever rationality arguments.

Or...

- Why bother? How important is NC for QM?