Completely Real?
A Critical Note on the Work of Colbeck & Renner

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Outline

1. What is the work of Colbeck and Renner about?
2. What have others concluded thus far about it?
3. The part that holds up: The Equiprobability Theorem.
4. A part that doesn't hold up: a single qubit.

Completeness of Quantum Mechanics

Colbeck & Renner 2015:

> *quantum theory is "maximally informative", i.e., there is no other compatible theory that gives improved predictions. Furthermore, any alternative maximally informative theory is necessarily equivalent to quantum theory. This means that the state a system has in such a theory is in one-to-one correspondence with its quantum-mechanical state (the wave function). In this sense, quantum theory is complete.*

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- **Completeness Theorem:** Impossibility to improve on predictions.
- **ψ-ontology Theorem:** States of systems determine their quantum state.

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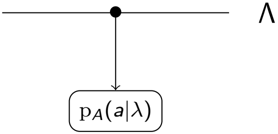
- Quantum states determine outcome probabilities: $\mathbb{P}_{|\psi\rangle}(A = a) = |\langle a|\psi\rangle|^2$.
- A more informative state λ also determines probabilities: $\mathrm{p}_\lambda(A = a)$.



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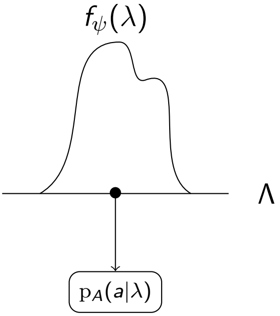

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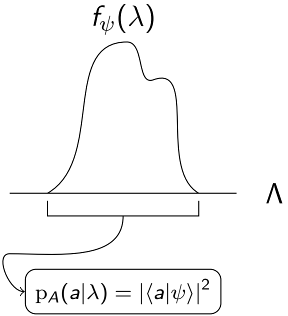

- The recovery is trivial if the λ-probabilities are equal to the quantum probabilities.
- If this holds for all A, the quantum state is *complete*.

Reality of the quantum state

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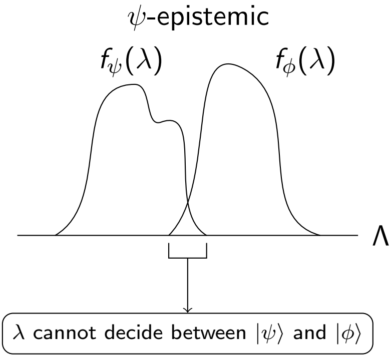
A sufficient condition for an epistemic interpretation of quantum states is that they can be represented by overlapping probability distributions.



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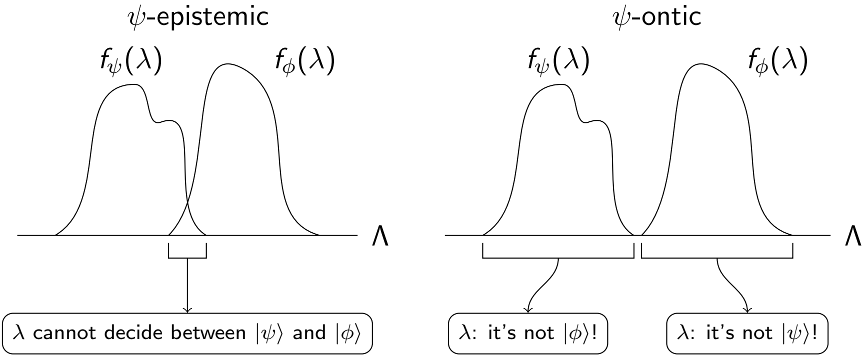

- **ψ-ontic:** Probability distributions for pure quantum states are pairwise non-overlapping.

Completeness of the Quantum State implies its Reality

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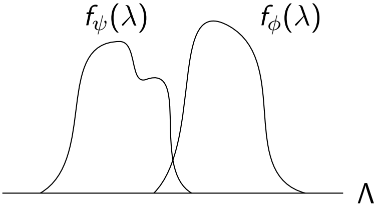
Proof: By Reductio ad Absurdum.

- Suppose there is overlap for ψ and ϕ:

ψ-epistemic



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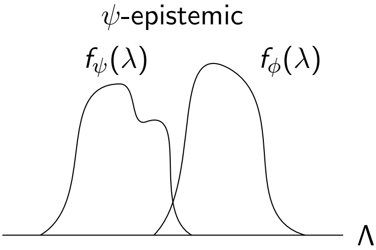

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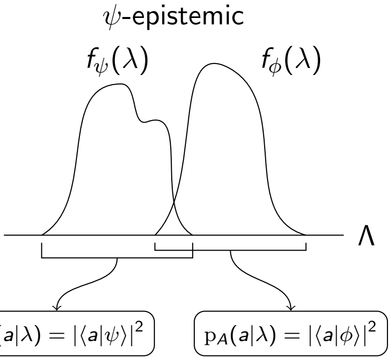

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- Completeness implies a contradiction on the overlap. □

Short Overview of Earlier Work

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- 2012 Colbeck and Renner's ψ-ontology theorem:

  Here we show, based only on the assumption that measurement settings can be chosen freely, that a system's wave function is in one-to-one correspondence with its elements of reality. This also eliminates the possibility that it can be interpreted subjectively.

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- 2016 Leegwater. Reformulation and proof of the Completeness theorem. Allegedly without obscure assumptions.

The Part That Holds Up

The Equiprobability Theorem

Proving Completeness: the Equiprobability Theorem

Equiprobability Theorem

For any ontic model that satisfies parameter independence. For local measurements on a qubit pair in the maximally entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right)$$

the λ-probabilities are equal to the quantum probabilities.

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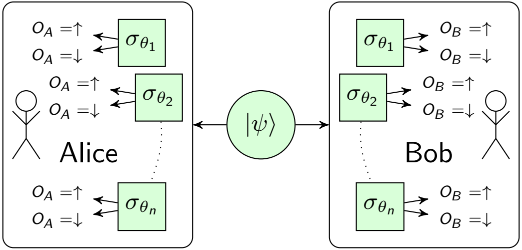

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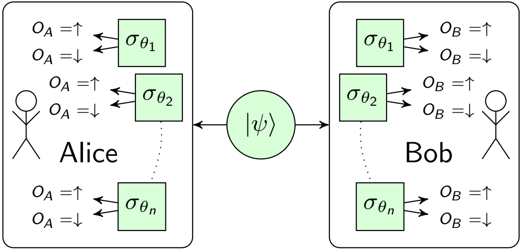

For any ontic model that satisfies Parameter Independence:

- $\mathrm{p}^A_{\theta_i}(\uparrow|\lambda) = \mathbb{P}_{|\psi\rangle}(\uparrow|\sigma_{\theta_i} \otimes \mathbb{1})$ almost surely w.r.t. $|\psi\rangle$.
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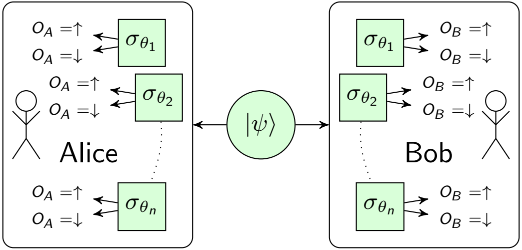

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$\Longrightarrow$ For local measurements on the singlet state the quantum predictions are complete.

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Then choose angles $\theta_1, \ldots, \theta_n$ between z and $-z$ and "chain up" Bell inequalities to obtain desired result.

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- arbitrary local measurements for arbitrary entangled states.
- non-local measurements.
- measurements on a single system.

(last three: **???**)

Measurements on a Single Qubit

Completeness for individual systems?

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Critical Note:

Colbeck and Renner do not succeed in giving such plausible arguments.

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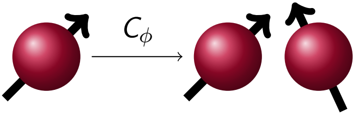

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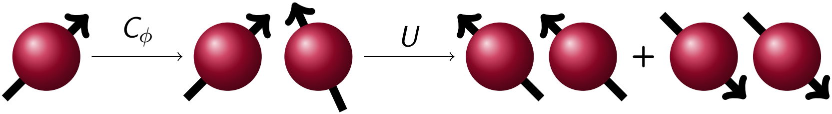
Structure of the Argument



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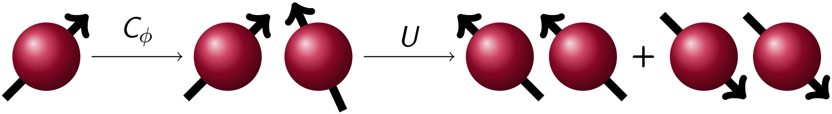

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- Unitarily transform to maximally entangled state:
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Because

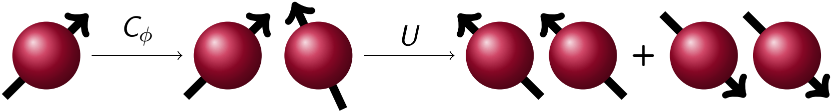
$$\mathbb{P}_{|\psi_0\rangle}(\uparrow|\sigma_z) = \mathbb{P}_{|\psi\rangle}(\uparrow|\mathbb{1}\otimes\sigma_z)$$

"the same relation holds when considering λ-probabilities"

$$\mathrm{p}_z^{A,|\psi_0\rangle}(\uparrow|\lambda) = \mathrm{p}_z^{B,|\psi\rangle}(\uparrow|\lambda)$$

Why would that follow?
What does it mean?

Formal motivation to warrant conclusion



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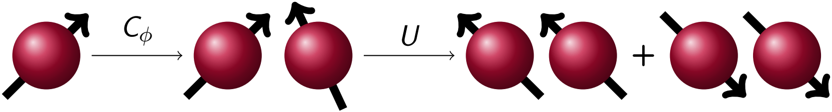

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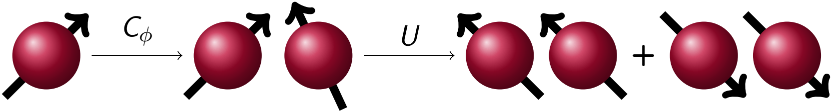
$$\Gamma_{C_\phi, U}(\lambda' | \lambda)$$

probability distribution over states for qubit pair conditional on initial state λ for single qubit.



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- Because $\mathbb{P}_{|\psi_0\rangle}(\uparrow|\sigma_z) = \mathbb{P}_{|\psi\rangle}(\uparrow|\mathbb{1}\otimes\sigma_z)$,

therefore $\boxed{\mathrm{p}_z^A(\uparrow|\lambda) = \int \mathrm{p}_z^B(\uparrow|\lambda')\Gamma_{C_\phi,U}(\lambda'|\lambda)\,\mathrm{d}\lambda'}$

almost surely w.r.t. $|\psi_0\rangle$.

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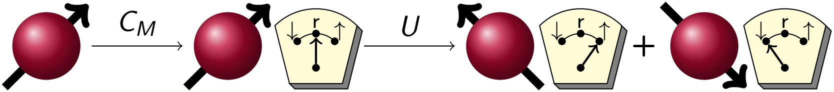

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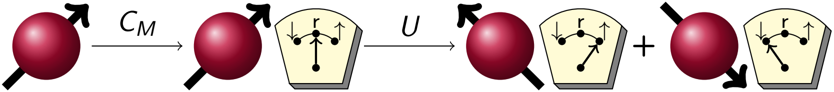

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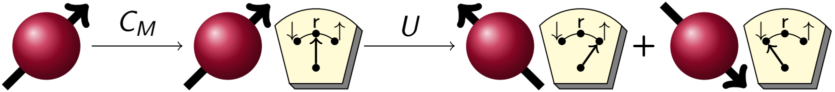

- For collapse theories the final state does not occur and the equiprobability theorem cannot be applied.
- For no-collapse theories application of equiprobability theorem requires:
  - Qubit and apparatus can be spatially separated.
  - Arbitrary other measurements can be made on the qubit.
  - Arbitrary *non-pointer* measurements can be made on the apparatus.